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EXHAUSTION OF A THIN FILM OF A NONLINEAR-VISCOUS FLUID FROM A SLOT WITH SLIPPING RELATIVE TO THE UNDERLYING SURFACE

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The problem of exhaustion of a thin film of a non-Newtonian fluid with a power rheological law from a slotted orifice is solved with account of film slipping relative to the underlying surface. By the method of group analysis with transformation of the parameters entering the problem, an asymptotic formula for the film profile is obtained and a law of motion of the film edge with small slipping is derived.

The problem of exhaustion of a nonlinear-viscous fluid from a slot was solved by Chugunov [1] under conditions of fluid adhesion to the substrate. Some applications, however, involve situations where it is necessary to consider slipping of the spreading film relative to the underlying surface. With account of slipping, Tonkonog et al. [2] considered the problem of free spreading of a drop of a non-Newtonian fluid over a horizontal surface and constructed its asymptotic solution for $\varepsilon \ll 1$ (small slipping). The paper [2] is based on using the invariance of the solution of the problem of drop spreading relative to a certain group of tensions, which transforms not only the independent variables and unknown function but also the parameter ε. This idea is also used in the present work to study the dynamics of the surface of a thin film of a nonlinear-viscous fluid excaping from a stationary slot and slipping relative to a horizontal substrate.

In accordance with [1, 3–5], the mathematical formulation of the problem posed can be written in dimensionless variables as

$$\frac{\partial l}{\partial t} = \frac{\partial q^\varepsilon}{\partial x}, \qquad t > 0, \qquad 0 < x < x_0(t); \tag{1}$$

$$q^\varepsilon = \operatorname{sign}\Big(\frac{\partial l}{\partial x}\Big)[l^2|\sigma|^n + \varepsilon l|\sigma|^m], \qquad \sigma = l l_x, \qquad n > m; \tag{2}$$

$$x_0(0) = 0, \qquad l(0,t) = 1, \quad t > 0, \qquad l(x_0(t),t) = 0, \quad q^\varepsilon(x_0(t),t) = 0. \tag{3}$$

Here l is the dimensionless thickness of the film, the front point $x_0(t)$ is unknown and has to be determined in the course of solving the problem, and $\varepsilon \ll 1$. Equation (1) is obtained using the known slipping model proposed by Kamb [5]. Since the case of interest is a monotonically decreasing solution of system (1)–(3), relation (2) for the flow can be rewritten in the following form:

$$q^\varepsilon = -l^2(-\sigma)^n - \varepsilon l(-\sigma)^m. \tag{4}$$

Note that the notation is borrowed from [2], and for $\varepsilon = 0$ problem (1)–(3) transforms into that solved in [1].

An infinitesimal operator corresponding to the group of transformations allowed by system (1)–(4) and transforming the parameter ε can be easily found:

$$Y = (n+1)t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} + (m-n)\varepsilon\,\frac{\partial}{\partial \varepsilon}.$$

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Invariants of this group are

$$I_1 = \xi = xt^{-\alpha}, \qquad I_2 = \eta = \varepsilon t^{\alpha(n-m)}, \qquad I_3 = l,$$

where $\alpha = 1/(n+1)$. Hence, the solution of problem (1)–(3) should be sought in the form

$$l = \psi(z,\eta), \qquad x_0(t) = \xi_0 t^{\alpha}, \qquad \xi_0 = g(\eta), \qquad z = \xi/g(\eta). \tag{5}$$

In the variables ψ, z, and η, system (1)–(3) becomes

$$\begin{gathered} \alpha g^{n+1}[z\psi_z - (n-m)\eta(\psi_\eta - zg_\eta g^{-1}\psi_z)] = q_z, \\ z = 0, \quad \psi = 1; \qquad z = 1, \quad \psi = 0, \quad q = 0, \end{gathered} \tag{6}$$

where $q = \psi^{n+2}(-\psi_z)^n + \eta\psi^{m+1}(-\psi_z)^m g^{n-m}$ and the subscripts z and η denote differentiation with respect to these variables. The solution of system (6) is sought with accuracy to $O(\eta^2)$ (note that η and ε are small over a finite range of t since $n > m$):

$$\psi(z,\eta) = V(z) + \eta U(z) + O(\eta^2), \qquad g(\eta) = a + b\eta + O(\eta^2). \tag{7}$$

Here $V(z)$ and $U(z)$ are unknown functions and a and b are unknown constants determining the sought functions l and x_0 with accuracy to $O(\varepsilon^2)$.

Using the definition of the function q, we easily find its expansion in powers of η: $q = q_0 + q_1\eta + O(\eta^2)$, where $q_0 = V^{n+2}(-V_z)^n$ and $q_1 = -nV^{n+2}(-V_z)^{n-1}U_z + (n+2)V^{n+1}(-V_z)^n U + a^{n-m}V^{m+1}(-V_z)^m$.

From (6) we obtain equations for determining V and U:

$$\alpha a^{n+1} zV_z = [V^{n+2}(-V_z)^n]_z, \qquad z = 0, \quad V = 1; \qquad z = 1, \quad V = 0, \quad q_0 = 0; \tag{8}$$

$$\begin{gathered} \alpha a^n\{b(2n+1-m)zV_z + a[zU_z - (n-m)U]\} = q_{1z}, \\ z = 0, \quad U = 0; \qquad z = 1, \quad U = 0, \quad q_1 = 0. \end{gathered} \tag{9}$$

Following [1], we assume

$$V = a^{\gamma}\psi_0(z), \qquad \gamma = (n+1)/(2n+1). \tag{10}$$

Then for ψ_0 we have the Cauchy problem

$$\alpha z\psi_{0z} = [\psi_0^{n+2}(-\psi_{0z})^n]_z, \quad z = 1, \quad \psi_0 = 0, \quad \psi_0^{n+2}(-\psi_{0z})^n = 0, \tag{11}$$

which does not contain the parameter a. This is important for further derivation. The solution of problem (11) is described in detail in [1]. It has the form

$$\psi_0 = C_n(1-z)^{\beta}[1 + d_1(1-z) + d_2(1-z)^2 + \ldots], \tag{12}$$

where $\beta = n/(2n+1)$ and $C_n = \beta^{-\beta}\alpha^{\gamma-\beta}$; the coefficients d_1 and d_2 are not used in what follows. Taking into account Eq. (10) and the first boundary condition of system (8), we can easily find the parameter a:

$$a = [\psi_0(0)]^{-1/\gamma}. \tag{13}$$

Note that Eq. (11) is invariant relative to a group of similarity with the operator $t\,\partial/\partial t + \alpha x\,\partial/\partial x$, hence, it allows a decrease in the order by the substitution

$$\psi_0 = z^{\gamma}\chi(z), \qquad \nu = z\chi'(z). \tag{14}$$

From (14), we obtain

$$\psi'_{0z} = z^{-\beta}\mu(z), \tag{15}$$

where $\mu = \nu + \gamma\chi$. With account of (14) and (15), Eq. (11) transforms to the first-order ordinary differential equation

$$\frac{d\nu}{d\chi} = \frac{P(\chi,\nu)}{Q(\chi,\nu)}. \tag{16}$$

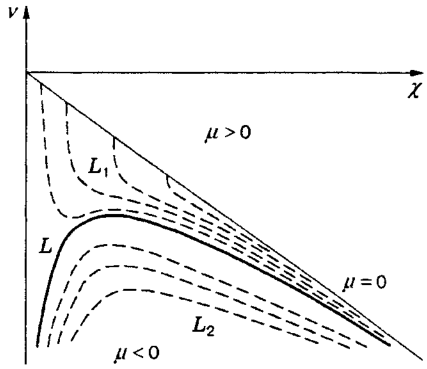

Fig. 1

Here $P(\chi,\nu) = \alpha + (n+2)\nu\chi^{n+1}(-\mu)^{n-1} - \beta\chi^{n+2}(-\mu)^{n-2}[(3n+2)\mu/n + (n+1)\nu]$; $Q(\chi,\nu) = n\nu\chi^{n+2}(-\mu)^{n-2}$.

By virtue of Eqs. (14) and (15), it is necessary to study the behavior of the solutions of Eq. (16) in the region $\chi \geqslant 0$, $\mu \leqslant 0$ (Fig. 1) to construct nonnegative, monotonically decreasing solutions of Eq. (11). In this region, there are two families L_1 and L_2 of the solutions of Eq. (16) (dashed curves) and the solution L separating them (solid curve). For curves of the family L_2 and the curve L, the straight line $\chi = 0$ is a vertical asymptote. The asymptotic curve $\nu \sim K\chi^{-(n+2)/n}$ is valid for curves of the family L for $\chi \to 0$ (the constant $K < 0$ acquires different values for different curves), and the asymptotic curve $\nu_L \sim -\alpha^{1/n}\chi^{-(n+1)/n}$ is obtained for the curve L as $\chi \to 0$. For $\chi \to \infty$, all solutions have a binomial asymptote $\nu \sim -\gamma\chi + M\chi^{-n\alpha}$ (the constant $M < 0$ takes different values for different curves). Thus, the straight line $\mu = 0$ is an inclined asymptote for all the curves mentioned.

By virtue of Eqs. (14) and (15), each solution of Eq. (16) that belongs to the region $\chi \geqslant 0$, $\mu \leqslant 0$ generates a one-parametric family of nonnegative, monotonically decreasing solutions of Eq. (11). Nevertheless, using the above-derived asymptotes, we can easily see that the only nonnegative, monotonically decreasing solution ψ_0 of Eq. (11) determined on the interval $0 \leqslant z \leqslant 1$, which satisfies both conditions (11) for $z = 1$, is generated by the solution $\nu = \nu_L(\chi)$ of Eq. (16). This solution is constructed as follows. From the second relation of (14), we obtain

$$\ln z = \int_0^{\chi} \frac{d\chi}{\nu_L(\chi)}, \qquad \chi \geqslant 0. \tag{17}$$

Equation (17) implicitly determines the function $\chi(z)$ on $0 \leqslant z \leqslant 1$ (we recall that $\nu_L(\chi) < 0$ on $0 < \chi < \infty$), and $\chi(0) = \infty$, $\chi(1) = 0$. As a result, the sought solution is determined using the first relation of (14); it is strictly greater than zero and finite at the point $z = 0$. This can be easily verified using the asymptotic curve $\nu_L(\chi)$ for $\chi \to \infty$, whence it follows that the constant a determined by formula (13) is positive and finite.

Relations (10), (12), and (13), with account of equalities (7), determine the first terms of expansions of the unknown functions ψ and g in powers of η.

To construct U, we write the expansions of the function V and the coefficients of Eq. (9) in powers of the binomial $1 - z$, confining ourselves to the main terms of these expansions for $z \to 1 - 0$:

$$V = C_n a^{\gamma}(1-z)^{\beta} + o[(1-z)^{\beta}], \qquad V_z = -\beta C_n a^{\gamma}(1-z)^{-\gamma} + o[(1-z)^{-\gamma}]; \tag{18}$$

$$\begin{gathered} nV^{n+2}(-V_z)^{n-1} = a^{n+1}\gamma^{-1}(1-z) + o(1-z), \\ (n+2)V^{n+1}(-V_z)^{n} = \alpha(n+2)a^{n+1} + o(1). \end{gathered} \tag{19}$$

The function U satisfies the inhomogeneous equation (9) for which the corresponding homogeneous equation can be written in the form

$$\alpha a^{n+1}[zw_z - (n-m)w] = [r(z)w_z + p(z)w]_z, \qquad (20)$$

where $r(z) = -nV^{n+2}(-V_z)^{n-1}$, $p(z) = (n+2)V^{n+1}(-V_z)^n$, and $r(z)$ and $p(z)$ are expanded in powers of $1-z$. With account of relations (19), the higher terms of these expansions are

$$r(z) \sim -a^{n+1}\gamma^{-1}(1-z), \qquad p(z) \sim \alpha(n+2)a^{n+1}.$$

Therefore, the solution of Eq. (20) should be sought in the form

$$w = (1-z)^\tau[1+o(1)], \qquad z \to 1-0.$$

From (20), we obtain the characteristic equation for τ

$$(n+1)\tau = -(2n+1)\tau^2,$$

from which we find $\tau_1 = -\gamma$ and $\tau_2 = 0$. Hence, we have two independent particular solutions of Eq. (20) with the asymptotic relations

$$w_1 = (1-z)^{-\gamma}[1+o(1)], \qquad w_2 = 1 + o(1).$$

The general solution of Eq. (20) has the form $w = Aw_1 + Bw_2$, where A and B are arbitrary constants.

The asymptotic behavior of the functions w_1 and w_2 shows that none of the solutions of homogeneous equation (20) satisfies the condition $U(1) = 0$. Thus, we can state that this condition determines the function $U(z)$ in a unique manner. We find the asymptotic behavior of $U(z)$ in the vicinity of the point $z = 1$. For this purpose, we rewrite Eq. (9) determining this function in the following form:

$$\alpha a^{n+1}[zU_z - (n-m)U] = [r(z)U_z + p(z)U]_z + f(z), \qquad (21)$$

Here $f(z) = f_1(z) + bf_2(z)$; $f_1(z) = a^{n-m}[V^{m+1}(-V_z)^m]_z$; $f_2(z) = -a^n\alpha(2n+1-m)zV_z$; the asymptotic behavior of the functions $r(z)$ and $p(z)$ for $z \to 1-0$ is described above.

Using expansions (18) and (19), we can easily obtain

$$f_1(z) \sim -C_n^{2m+1}\beta^m(1-\chi)a^{n+\chi}(1-z)^{-\chi}, \qquad f_2(z) \sim C_n\alpha\beta(2n+1-m)a^{n+\gamma}(1-z)^{-\gamma},$$
$$\chi = (n+m+1)/(2n+1). \qquad (22)$$

The solution of Eq. (21) should be sought in the form

$$U = U_1(z) + bU_2(z), \qquad (23)$$

where U_i is the solution of Eq. (21) for $f(z) = f_i(z)$, $i = 1, 2$. If $z \to 1-0$, we obtain

$$U_i(z) = C_{i0}(1-z)^{r_i}, \qquad i = 1, 2. \qquad (24)$$

Substituting (23) and (24) into (21), with account of (22), we find

$$r_1 = 1-\chi, \qquad r_2 = 1-\gamma,$$

$$C_{10} = -C_n^{2m+1}a^{\chi-1}\beta^m\alpha^{-1}(2n-m+1)^{-1}, \qquad C_{20} = C_n a^{\gamma-1}(2n+1-m)/(3n+2).$$

Obviously, for $z \to 1-0$ and $m > 0$, only the highest term of the first component remains in the asymptotic relation in the right side of Eq. (23), i.e.,

$$U \sim C_{10}(1-z)^{1-\chi}, \qquad (25)$$

and hence, we have

$$q_1 \sim B(1-z)^{1-\chi}, \qquad B = C_n^{2m+1}a^{n+\chi}\beta^m(2n+1-m)^{-1}. \qquad (26)$$

Since $n > m$, we have $1-\chi > 0$, and it follows from (25) and (26) that the boundary conditions are satisfied for the function U at the point $z = 1$. We have to find the parameter b using the boundary condition at the point $z = 0$. Taking into account that $U(0) = 0$, from (23) we find $b = -U_1(0)/U_2(0)$.

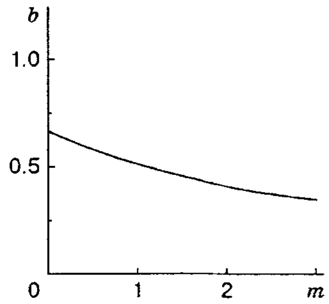

Fig. 2

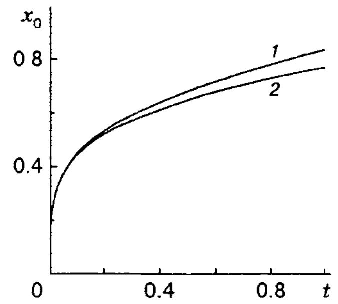

Fig. 3

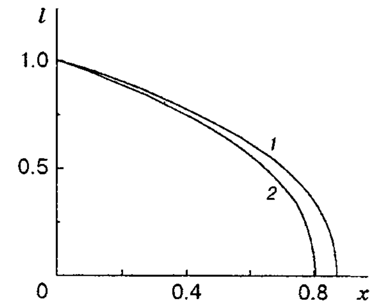

Fig. 4

Similarly to [2] the dependence of the parameter b on a can be easily established. Indeed, according to the definition of the functions f_1 and f_2 and formula (10), we obtain

$$f_1(z) = a^{n+\chi}[\psi_0^{m+1}(-\psi_{0z}^m)]_z, \qquad f_2(z) = -\alpha a^{n+\gamma}(2n+1-m)z\psi_{0z}.$$

Therefore, we have

$$U_1(z) = a^{\chi-1}U_{11}(z), \qquad U_2(z) = a^{\gamma-1}U_{21}(z), \tag{27}$$

where $U_{11}(z)$ and $U_{21}(z)$ are the solutions of Eq. (21) for $a = 1$. Hence, we obtain

$$b = -a^{\chi-\gamma}U_{11}(0)/U_{21}(0). \tag{28}$$

Finally, we formulate the algorithm for solving the posed problem.

1. From system (11), which is easily reduced to the Cauchy problem, we find $\psi_0(z)$:

$$\psi_{0z} = -s^{1/n}\psi_0^{-(n+2)/n}, \quad s_z = -\alpha z s^{1/n}\psi_0^{-(n+2)/n}, \quad z = 1, \quad \psi_0 = 0, \quad s = 0.$$

   In numerical solution of the Cauchy problem, the initial conditions are imposed at a point close to the point $z = 1$ using the asymptotic formula (18).
2. From formula (13), we find the parameter a.
3. From the solution of two Cauchy problems

$$U_z = [S - p(z)U]/r(z),$$

$$S_z = \alpha a^{n+1}\{zS - U[zp(z) + r(z)(n-m)]\}/r(z) - f_i(z), \qquad i = 1, 2, \quad z < 1;$$

$$U = 0, \qquad S = 0, \qquad a = 1, \qquad z = 1$$

   we find the functions $U_{11}(z)$ and $U_{21}(z)$.
4. The parameter b is found from formula (28), and the functions U_1 and U_2 are built using formulas (27).
5. The values of the functions V and U are calculated from formulas (10) and (23).
6. From the calculated values of V, U, a, and b, the unknown functions $l(x,t)$ and $x_0(t)$ are determined with accuracy to terms of order $O(\varepsilon^2)$ using relations (5) and (7).

The results of calculations conducted using the algorithm proposed are plotted in Figs. 2–4 (it was assumed that $\varepsilon = 0.1$. Figure 2 shows the coefficient b [see (28)] as a function of the power exponent in Kamb's law. As m increases, the coefficient b decreases, hence, the effect of slipping on the motion of the film edge decreases. In addition, as $m \to n$, the effect of slipping on the law of motion of the film edge becomes independent of the time factor. Figure 3 shows the coordinate of the film edge x_0 versus the time t with account (curve 1) and without account (curve 2) of slipping (it was assumed that $m = 0.5$ and $n = 3$ in the calculations). As should be expected, the velocity of the film edge with slipping is greater than without it. Thus, the profile of the film slipping relative to the substrate (curve 1 in Fig. 4, $t = 1$) is always located higher than the profile of the film spreading under condition of its adhesion to the substrate (curve 2 in Fig. 4).

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REFERENCES

1. V. A. Chugunov, "Solution of two self-similar problems of the flow of a subisothermal glacier," in: *Studies in Applied Mathematics* [in Russian], Izd. Kazan. Univ., Kazan', No. 11, Part 2, 105–110 (1984).
2. S. L. Tonkonog, V. A. Chugunov, and L. D. Eskin, "Methods of group analysis in the problem of spreading of a thin film of a nonlinear-viscous fluid with slipping," *Appl. Mat. Mekh.*, **58**, No. 4, 63–69 (1994).
3. A. N. Salamatin, V. A. Chugunov, and A. B. Mazo, "Numerical and invariant solutions of the problems of dynamics of a subisothermal glacier in the one-dimensional approximation," in: *Problems of Mechanics of Natural Processes* [in Russian], Izd. Mosk. Univ., Moscow (1983), pp. 82–96.
4. C. Ritz, "Un modele thermo-mecanique d'evolution pour le bassin glaciaire antarctique Vostok-Clacier Byrd: sensibilite aux valeurs des parametres mal connus," These Doctorat d'Etat, Centre Nat. de la Rech. Sci., Lab. de Glaciol. et Geophys. de l'Environnement, Grenobl (1992), p. 377.
5. B. Kamb, "Sliding motion of glaciers: theory and observation," *Rev. Geophys. Space Phys.*, **8**, No. 4, 673–728 (1970).